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Extra diagnosability and good-neighbor diagnosability of n-dimensional alternating group graph AG_n under the PMC model

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ABSTRACT

The h-extra diagnosability and g-good-neighbor diagnosability are two important diagnostic strategies at system-level that can significantly enhance the system's self-diagnosing capability. The h-extra diagnosability ensures that every component of the system after removing a set of faulty vertices has at least $h+1$ vertices. The g-good-neighbor diagnosability guarantees that after removing some faulty vertices, every vertex in the remaining system has at least g neighbors. In this paper, we analyze the extra diagnosability and good-neighbor diagnosability in a well-known n-dimensional alternating group graph AG_n proposed for multiprocessor systems under the PMC model. We first establish that the 1-extra diagnosability of AG_n $(n \geq 5)$ is $4n-10$. Then we prove that the 2-extra diagnosability of AG_n $(n \geq 5)$ is $6n-17$. Next, we address that the 3-extra diagnosability of AG_n $(n \geq 5)$ is $8n-25$. Finally, we obtain that the g-restricted connectivity and the g-good-neighbor diagnosability of AG_n $(n \geq 5)$ are $(2g+2)n-2^{g+2}-4+g$ and $(2g+2)n-2^{g+2}-4+2g$ for $1 \leq g \leq 2$, respectively.

1. Introduction

Processor failure is inevitable with the increasing sizes of multiprocessor systems. To ensure the stable running of the systems, we must find out the faulty processors to repair or replace them to maintain the high availability of the system. Therefore, fault diagnosis of interconnection networks has become increasingly important due to the rapid development of multiprocessor systems.

The process of identifying faulty processors by analyzing the outcome of mutual tests among processors is called *system-level diagnosis*. Preparata, Metze, and Chien [23] proposed a foundation of system diagnostic model—*Preparata, Metze, and Chien* (*PMC*) *model*. Under the PMC model, all tests are performed between two adjacent processors, and a test result is reliable (resp., unreliable) if the tester is fault-free (resp., faulty). Obviously, a faulty processor cannot yield correct test result.

Zhang and Yang [35] proposed the *h-extra diagnosability*, which is a generalization of conditional diagnosability [1,19,20]. The h-extra diagnosability is defined under the assumption that every component of the system after removing a set of

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faulty vertices has more than h vertices. Xu et al. [33] explored the extra diagnosability of arrangement graphs. Han and Wang [7] gave the h-extra conditional diagnosability of n-folded hypercubes for $0 \le h \le n-4$. Wang et al. [28] established the 3-extra diagnosability of bubble-sort star graph networks. Wang and Yang [29] proposed the 2-extra diagnosability of n-dimensional alternating group network AN_n, which is different from AG_n. In 2015, Zhang and Yang [35] gave the h-extra diagnosability of n-hypercubes for $n-4 \le h \le 3n-7$. Zhu and Zhang [36] extended their result to $n-3 \le h \le 3n-7$ under the PMC model in 2016.

In 2010, Peng et al. [16,22] first proposed the notion of *g-good-neighbor diagnosability*, defined as the maximum number t such that a graph G remains t-diagnosable under the condition that every healthy vertex has at least g fault-free neighbors. It is formulated into the question: for a network $G(V, E)$, if the faulty set $F \subset V(G)$ satisfies the condition that each vertex v in $G-F$ has at least g good neighbors, then what is the maximum t such that G is still t-diagnosable? The answer to this question for a network of arbitrary topology is obviously intractable. The g-good-neighbor diagnosability of hypercubes [22] is proposed by considering the known g-restricted connectivity and the size of the g-dimensional hypercube (Q_g). Yuan et al. [34] gave the g-good-neighbor diagnosability of k-ary n-cubes. Xu et al. [32] also proposed the g-good-neighbor diagnosability of (n, k)-star graphs by combining the known g-restricted connectivity with the size of the $(g+1)$-dimensional complete graph (K_{g+1}). Gu et al. [6] gave a short note on the $\{1, 2\}$-good-neighbor diagnosability of balanced hypercubes. Lin et al. [18] established the g-good-neighbor diagnosability of arrangement graphs by exploring the known g-restricted connectivity and the size of K_{g+1}. Wei and Xu [30] also gave g-good-neighbor diagnosability of (n, k)-star networks. Li and Lu [15] proposed g-good-neighbor conditional diagnosability of star graphs. In 2017, Jirimutu and Wang [11] established the 1-good-neighbor diagnosability of alternating group networks AN_n. In 2018, Wei and Xu [31] explored the $\{1, 2\}$-good-neighbor conditional diagnosability of regular graphs. Also, Lin et al. [17] established the relationship between g-restricted connectivity and g-good-neighbor fault diagnosability of general triangle-free regular networks.

In 1993, Jwo et al. [12] first proposed the *n-dimensional alternating group graph* AG_n as a topology of interconnection network for multiprocessor systems. The n-dimensional alternating group graph AG_n possesses sufficient amount of good properties including cycle-embedding [2,25], and small diameter [12]. Moreover, the alternating group graph is not only pancyclic and hamiltonian-connected [3], but also panconnected [9]. It also has a fault-free longest path [24] and vertex pancyclicity [26]. Furthermore, Lin et al. [21] established the extra fault tolerance and conditional diagnosability of AG_n in 2015. There is another type of interconnection network based on alternating group called the n-dimensional alternating group network AN_n [10], which is different from the n-dimensional alternating group graph AG_n [12] investigated in this paper. Both of the AG_n and AN_n are Cayley graphs, but with different generating sets. Consequently, they have distinct adjacency manners. Roughly speaking, the edges of AG_n are generated by $(12i)$ and $(1i2)$ for $i = 3, \ldots, n$, while the edges of AN_n are generated by (123), (132) and $(12)(3i)$ for $i \in \{4, \ldots, n\}$.

We were motivated by the recent research on the g-extra diagnosability of the (n, k)-arrangement graph [33] under the PMC model, and the 2-extra diagnosability of alternating group network AN_n [29]. Moreover, Wang and Ren [27] investigated the 2-extra diagnosability of alternating group graph AG_n. We extend this result to $\{1, 2, 3\}$-extra diagnosability of AG_n. First, we establish that the 1-extra diagnosability of AG_n $(n \ge 5)$ under the PMC model is $4n-10$. Then we prove that the 2-extra diagnosability of AG_n $(n \ge 5)$ under the PMC model is $6n-17$. Next, we address that the 3-extra diagnosability of AG_n $(n \ge 5)$ under the PMC model is $8n-25$. Finally, motivated by the g-good-neighbor diagnosability of the (n, k)-arrangement graph [18] under the PMC model, we tackle that the g-restricted connectivity and the g-good-neighbor diagnosability of AG_n $(n \ge 5)$ are $(2g+2)n - 2^{g+2} - 4 + g$ and $(2g+2)n - 2^{g+2} - 4 + 2g$ for $1 \le g \le 2$, respectively.

Organization. The remainder of this paper is organized as follows. Section 2 introduces some preliminaries for this paper. The h-extra diagnosability of AG_n is addressed under the PMC model in Section 3. Section 4 proposes the g-restricted connectivity and the g-good-neighbor diagnosability of AG_n under the PMC model. Section 5 concludes this paper.

2. Preliminaries

We present some necessary terms in the graph theory in this section. Then we propose the definitions of extra diagnosability and good-neighbor diagnosability under the PMC model. Moreover, we show the definition of an n-dimensional alternating group graph AG_n.

2.1. Terminologies and notations

The notation $G = (V(G), E(G))$ represents an interconnection network, where $V(G)$ is the vertex-set and $E(G)$ is the edge-set. Also, the terms $|V(G)|$ and $|E(G)|$ denote the numbers of vertices and edges of G, respectively.

A subgraph H of G, denoted by $H \subseteq G$, is a graph in which $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph of G induced by a subset S of $V(G)$, denoted by $G[S]$, is the graph with the vertex-set S and the edge-set $\{uv \mid uv \in E(G), u, v \in S\}$. Let $G_1, G_2, \ldots, G_m$ be m subgraphs of G, we set $\bigcup_{i=1}^{m} G_i = G[\bigcup_{i=1}^{m} V(G_i)]$ and $\bigcap_{i=1}^{m} G_i = G[\bigcap_{i=1}^{m} V(G_i)]$. When G is a graph and $F \subseteq V(G)$, $G - F$ denotes a graph obtained by removing all vertices in F from G and deleting those edges with at least one end-vertex in F, simultaneously. Let M and N be any two distinct sets of $V(G)$. Let $M \triangle N = (M - N) \cup (N - M) = \{x \mid x \in M \cup N, x \notin M \cap N\}$. We also denote $E[M, N]$ to be the set of all edges between M and N.

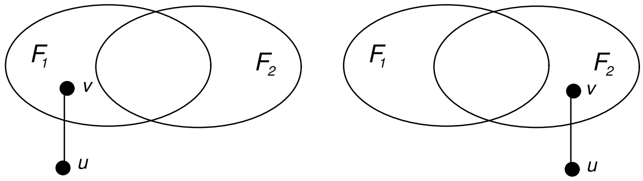

Fig. 1. An illustration of a distinguishable pair (F_1, F_2) under the PMC model.

The neighborhood $N_G(v)$ of vertex v in G is $\{u \in V(G) \mid uv \in E(G)\}$. Let $N_G[v] = N_G(v) \cup \{v\}$. We define $N_G(S) = \{v \in V(G) - S \mid \exists u \in S, uv \in E(G)\} = (\bigcup_{u\in S} N(u)) - S$. Let $N_G[S] = N_G(S) \cup S$. We also denote by $|N(v)|$ the degree $d(v)$ of vertex v. Let $\delta(G) = \min\{d(u) \mid u \in V(G)\}$ be the minimum degree of G. A *path* in a graph is a sequence of distinct vertices so that there are edges joining consecutive vertices, with the length being the number of vertices in the sequence minus 1. A *cycle* is a path of length at least 3 where there is an edge joining the first and last vertices. A path (or cycle) of length k is called a *k-path* (or *k-cycle*). We use $d(u, v)$ to denote the *distance* between u and v, the length of a shortest path between u and v in G.

2.2. Extra diagnosability and good-neighbor diagnosability

The *fault-set* of G is the set of all faulty vertices of G. We adopt the reasonable assumption that a vertex is capable of testing another vertex if and only if there is a direct communication link between them.

A *test assignment* for a multiprocessor system G is a collection of tests (u, v) for some adjacent pairs of vertices. If uv is any edge of $E(G)$, then u and v can test each other. Throughout this paper, we assume that each vertex tests another vertex whenever there is an edge between them and all these tests are gathered in a test assignment.

This paper adopts the PMC model [23], which assumes that a fault-free vertex can correctly evaluate all neighbors, whereas the outcome of a test conducted by a faulty vertex is completely unreliable.

Under the PMC model, Lai et al. [13] provided a necessary and sufficient condition to test whether a pair of faulty sets is distinguishable.

Lemma 1. [13,23] *Let $G = (V(G), E(G))$ be a multiprocessor system. Then for any two sets $F_1, F_2 \subseteq V(G)$ with $F_1 \neq F_2$, (F_1, F_2) is a distinguishable pair under the PMC model if and only if there exists a vertex $u \in V(G) - (F_1 \cup F_2)$, which is adjacent to a vertex $v \in F_1 \triangle F_2$ (see Fig. 1).*

Since it is impossible that all neighbors of some processors are simultaneously faulty, Zhang and Yang [35] proposed the *h-extra diagnosability*.

Definition 1. [35] (1) A set F is called an *h-extra vertex-set* if each remaining component in $G - F$ has at least $h + 1$ vertices.

(2) A system G is *h-extra t-diagnosable* under the PMC model if and only if for each pair of distinct faulty h-extra vertex-sets $F_1, F_2 \subseteq V(G)$ such that $|F_1| \leq t$ and $|F_2| \leq t$, F_1 and F_2 are distinguishable under the PMC model.

(3) The *h-extra diagnosability* of G under the PMC model, denoted as $\widetilde{t_h^p}(G)$, is the maximum value of t such that G is h-extra t-diagnosable.

Motivated by the idea of conditional diagnosability[13], Peng et al.[22] proposed the g-good-neighbor diagnosability with the condition that every fault-free vertex in the system has at least g fault-free neighbors.

Definition 2. [22] (1) A set F is called a g-good-neighbor vertex-set of G if $|N(v) \cap (V(G) - F)| \geq g$ for every vertex v in $V(G) - F$.

(2) A system $G = (V(G), E(G))$ is g-good-neighbor t-diagnosable if each distinct pair of g-good-neighbor vertex-sets F_1 and F_2 of $V(G)$ with $|F_1| \leq t$, $|F_2| \leq t$ are distinguishable.

(3) The g-good-neighbor diagnosability $t_g^p(G)$ of a graph G under the PMC model is the maximum value of t such that G is g-good-neighbor t-diagnosable.

2.3. Alternating group graph

Let $\langle n\rangle = \{1, \ldots, n-1, n\}$ and let $\zeta = \zeta_1\zeta_2\ldots\zeta_n$ be a permutation of elements in $\langle n\rangle$ where $\zeta_\alpha \in \langle n\rangle$ for $1 \leq \alpha \leq n$ and $\zeta_\alpha \neq \zeta_\beta$ for $1 \leq \alpha \neq \beta \leq n$. A pair of elements ζ_α and ζ_β is said to be an inversion of ζ if $\zeta_\alpha < \zeta_\beta$ whenever $1 \leq \beta < \alpha \leq n$. An even permutation is a permutation with an even number of inversions. Let A_n denote the set of all even permutations over $\langle n\rangle$.

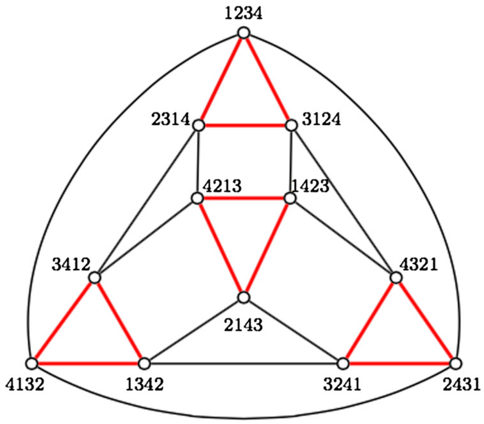

Fig. 2. An illustration of a 4-alternating group graph.

Definition 3. [12] The n-dimensional alternating group graph AG_n consists of vertex-set $V(AG_n) = A_n$ and edge-set $E(AG_n) = \{\zeta\eta \mid \eta$ is the permutation obtained from ζ by rotating the symbols in positions 1, 2, and α from left to right or from right to left for some $\alpha \in \{3, \ldots, n-1, n\}\}$.

It can be seen that AG_n is regular of degree $2n-4$, $|V(AG_n)| = \frac{n!}{2}$ and $|E(AG_n)| = \frac{(n-2)n!}{2}$. Fig. 2 describes an example of AG_4.

Denote by A_n^α ($n \geq 3$ and $1 \leq \alpha \leq n$) the subset of A_n consisting of all even permutations with α in the n-th position, and denote by AG_n^α the subgraph of AG_n induced by A_n^α. It implies that AG_n^α is isomorphic to AG_{n-1} for any $\alpha \in \langle n \rangle$. For convenience, let $AG_n^I = \bigcup_{i \in I} AG_n^i$ for any subset $I \in \langle n \rangle$ throughout this paper. The n-dimensional alternating group graph is composed of n disjoint copies of $(n-1)$-dimensional alternating group graphs such that AG_n^α connects AG_n^β $(1 \leq \alpha \neq \beta \leq n)$ by $(n-2)!$ disjointed edges due to the hierarchical structure. These $(n-2)!$ disjointed edges are called *external edges* with the form $\zeta\eta$ where $\zeta = \gamma\beta\cdots\alpha, \eta = \alpha\gamma\cdots\beta$ or $\zeta = \beta\gamma\cdots\alpha, \eta = \gamma\alpha\cdots\beta$ for $\gamma \in \langle n \rangle - \{\alpha, \beta\}$. Let $E_n^{\alpha,\beta}(AG_n)$ be the set of edges in AG_n connecting AG_n^α and AG_n^β for $1 \leq \alpha \neq \beta \leq n$. On the contrast, the edges connecting vertices in the same subgraph are called *internal edges*. In particular, for each internal edge $\zeta\eta$ with $\zeta = \gamma\beta\cdots\delta\cdots\alpha$ and $\eta = \beta\delta\cdots\gamma\cdots\alpha$ in AG_n^α, there are two adjacent vertices $\zeta' = \alpha\gamma\cdots\delta\cdots\beta$ and $\eta' = \delta\alpha\cdots\gamma\cdots\beta$ in AG_n^β such that $\{\zeta, \zeta', \eta', \eta\}$ is a 4-cycle in AG_n. This property is called the 4-cycle structure of $\zeta\eta$. Note that the vertices ζ' and η' are uniquely determined by the 4-cycle structure of $\zeta\eta$. Moreover, every vertex $\zeta \in V(AG_n^\alpha)$ lies on exactly $2n-6$ internal edges and two external edges. Furthermore, the two end-vertices, connecting a vertex via these two external edges are in distinct induced subgraphs.

Lemma 2. *An n-dimensional alternating group graph AG_n has the following combinatorial properties.*

(1) [12] *AG_n is $(2n-4)$-regular and $\kappa(AG_n) = \delta(AG_n) = 2n-4$ for $n \geq 3$.*

(2) [12] *The two external neighbors of every vertex of AG_n^i are in distinct $(n-1)$-dimensional subgraphs for $n \geq 4$.*

(3) [12] *There are $(n-2)!$ disjoint cross edges between every two distinct AG_n^i and AG_n^j for $1 \leq i \neq j \leq n$ and $n \geq 4$.*

(4) [8,21] *Let ζ, η be any two vertices of AG_n. If $\zeta\eta \notin E(AG_n)$, then $|N(\zeta) \cap N(\eta)| \leq 2$. If $\zeta\eta \in E(AG_n)$, then $|N(\zeta) \cap N(\eta)| = 1$.*

Latifi et al. [14] introduced the g-restricted connectivity, which implies that any vertex has no fewer than g neighbors in any remaining component.

Definition 4. [5] Given a graph G and two nonnegative integers h and g.

(1) An h-extra vertex-cut of G is the set of vertices whose deletion disconnects G and leaves each remaining component with at least $h+1$ vertices. The *h-extra connectivity* of G, denoted by $\kappa_0^{(h)}(G)$, is the minimum cardinality of h-extra vertex-cuts.

(2) An g-restricted vertex-cut of G is the set of vertices whose deletion disconnects G and the minimum degree of each remaining component is no less than g. The *g-restricted connectivity* of G, denoted by $\kappa^g(G)$, is the minimum cardinality of g-restricted vertex-cuts.

Lemma 3. [21] *Let AG_n $(n \geq 5)$ be the n-dimensional alternating group graph.*

(1) *The 1-extra connectivity of AG_n is $\kappa_0^{(1)}(AG_n) = 4n-11$. Furthermore, let $S = uv$ be an edge in AG_n such that $u = 1234\cdots i\cdots n$ and $v = 2431\cdots i\cdots n$. It can be deduced that $N(V(S))$ is a 1-extra vertex-cut of AG_n.*

(2) *The 2-extra connectivity of AG_n is $\kappa_0^{(2)}(AG_n) = 6n-19$. Furthermore, let $S = P_2 = \{u, v, w\}$ be a 2-path of AG_n such that $u = 1234\cdots i\cdots n$, $v = 2431\cdots i\cdots n$ and $w = 3124\cdots i\cdots n$. It can be deduced that $N(V(S))$ is a 2-extra vertex-cut of AG_n.*

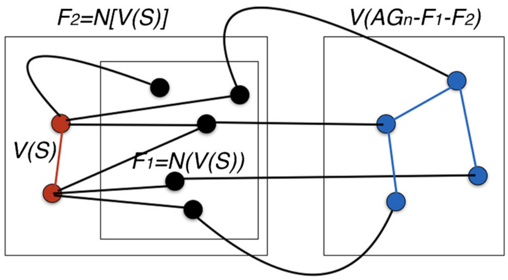

Fig. 3. An illustration that there is no edge between $F_1 \triangle F_2$ and $V(AG_n - F_1 - F_2)$ when $|V(S)| = 2$.

(3) *The 3-extra connectivity of AG_n is $\kappa_0^{(3)}(AG_n) = 8n - 28$. Furthermore, let $S = C_4 = \{u, v, w, x\}$ be a 4-cycle in AG_n such that $u = 1234 \cdots i \cdots n$, $v = 3124 \cdots i \cdots n$, $w = 4321 \cdots i \cdots n$ and $x = 2431 \cdots i \cdots n$. It can be deduced that $N(V(S))$ is a 3-extra vertex-cut of AG_n.*

3. Extra diagnosability of AG_n under the PMC model

Wang and Yang [29] proposed the 2-extra diagnosability of AN_n under the PMC model. In this section, we determine the h-extra diagnosability $\widetilde{t_h}^p(AG_n)$ of AG_n under the PMC model. First, we propose an upper bound of $\widetilde{t_h}^p(AG_n)$ by the construction method. Then the lower bound of the $\widetilde{t_h}^p(AG_n)$ is presented by the h-extra connectivity [21] listed as follows.

Theorem 1. *Let AG_n $(n \geq 5)$ be the n-dimensional alternating group graph. The 1-extra diagnosability of AG_n under the PMC model is $\widetilde{t_1}^p(AG_n) = 4n - 10$.*

Proof. We first prove that $\widetilde{t_1}^p(AG_n) \leq 4n - 10$. Let $S = uv$ be an edge in AG_n $(n \geq 5)$ with $u = 1234 \cdots i \cdots n$ and $v = 2431 \cdots i \cdots n$. We have

$$N(\{u, v\}) = (\bigcup_{5 \leq i \leq n} N_i) \bigcup N_0,$$

where

$$N_i = \{2i34 \cdots 1 \cdots (n-1)n, i134 \cdots 2 \cdots (n-1)n, \\ i231 \cdots 4 \cdots (n-1)n, 4i31 \cdots 2 \cdots (n-1)n\}$$

for $5 \leq i \leq n$, and

$$N_0 = \{2314 \cdots (n-1)n, 4132 \cdots (n-1)n, 3241 \cdots (n-1)n, \\ 3124 \cdots (n-1)n, 4321 \cdots (n-1)n\}.$$

Also, we have $|N(V(S))| = |N(\{u, v\})| = 4n - 11$. Let $F_1 = N(V(S))$ and $F_2 = N[V(S)]$. We have

$$|F_1| = 4n - 11, |F_2| = 4n - 9$$

and each component in $AG_n - N[V(S)]$ has more than one vertex by Lemma 3 (1). Hence, $N(V(S))$ is a 1-extra vertex-cut of AG_n. Therefore,

$$|F_1|, |F_2| \leq 4n - 9$$

and both F_1 and F_2 are 1-extra vertex-sets of AG_n.

Since $F_1 \triangle F_2 = V(S)$, there is no edge between $F_1 \triangle F_2$ and $V(AG_n - F_1 - F_2)$ (see Fig. 3). By Lemma 1, F_1 and F_2 are indistinguishable under the PMC model. By Definition 1, $\widetilde{t_1}^p(AG_n) \leq 4n - 10$.

Next, we prove the lower bound of 1-extra diagnosability of AG_n by contradiction. Suppose that $\widetilde{t_1}^p(AG_n) \leq 4n - 11$. Let F_1 and F_2 be two distinct 1-extra vertex-sets of AG_n such that (F_1, F_2) is an indistinguishable pair with $|F_1|, |F_2| \leq 4n - 10$.

For $n \geq 5$, we have

$$|V(AG_n)| - |F_1 \cup F_2| \geq \frac{n!}{2} - 2(4n - 10) > 0.$$

Thus,

$$V(AG_n - F_1 - F_2) \neq \emptyset.$$

By Lemma 1 and the fact that F_1 and F_2 are indistinguishable, $E[F_1 \triangle F_2, V(AG_n - F_1 - F_2)] = \emptyset$. Hence, the vertices of $F_1 \triangle F_2$ have no neighbors outside of $F_1 \cup F_2$ and the vertices of $V(AG_n - F_1 - F_2)$ have no neighbors in $F_1 \triangle F_2$. By the

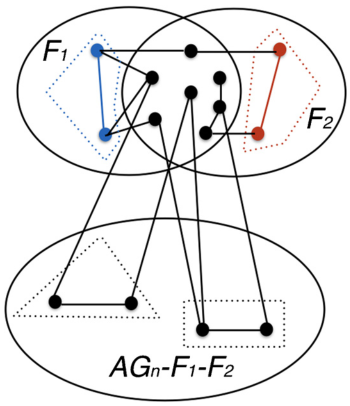

Fig. 4. An illustration that $AG_n - (F_1 \cap F_2)$ is disconnected and every component contains more than one vertex.

fact that F_1 and F_2 are two distinct 1-extra vertex-sets of AG_n, every component in $AG_n - F_1$ and $AG_n - F_2$ contains more than one vertex. When $F_1 \cap F_2$ is deleted, $AG_n - (F_1 \cap F_2)$ is disconnected and every component contains more than one vertex (see Fig. 4). By Definition 4 (1), $F_1 \cap F_2$ is a 1-extra vertex-cut of AG_n. By Lemma 3 (1), we have that

$$|F_1 \cap F_2| \geq 4n - 11.$$

Since $F_1 \neq F_2$, by the symmetry of AG_n, assume that $F_1 - F_2 \neq \emptyset$. Since F_2 is a 1-extra vertex-set of AG_n, every component of $AG_n - F_2$ contains more than one vertex. Since $E[F_1 \triangle F_2, V(AG_n - F_1 - F_2)] = \emptyset$, every component of $AG_n[F_1 - F_2]$ contains more than one vertex. That is $|F_1 - F_2| \geq 2$. Then

$$\begin{aligned}|F_1| &= |F_1 - F_2| + |F_1 \cap F_2| \\ &\geq 2 + 4n - 11 \\ &= 4n - 9,\end{aligned}$$

which contradicts that $|F_1| \leq 4n - 10$.

Hence, $\widetilde{t_1}^p(AG_n) = 4n - 10$. □

Theorem 2. *Let AG_n ($n \geq 5$) be the n-dimensional alternating group graph. The 2-extra diagnosability of AG_n under the PMC model is $\widetilde{t_2}^p(AG_n) = 6n - 17$.*

Proof. We first prove that $\widetilde{t_2}^p(AG_n) \leq 6n - 17$. Let $S = P_2 = \{u, v, w\}$ be a 2-path in AG_n ($n \geq 5$) with $u = 1234 \cdots i \cdots n$, $v = 2431 \cdots i \cdots n$ and $w = 3124 \cdots i \cdots n$. Obviously, P_2 is in AG_n^n. Furthermore, we have

$$N(V(S)) = (\bigcup_{5 \leq i \leq n} N_i) \bigcup N_0,$$

where

$$\begin{aligned}N_i = \{&2i34 \cdots 1 \cdots (n-1)n, i134 \cdots 2 \cdots (n-1)n, i231 \cdots 4 \cdots (n-1)n, \\ &4i31 \cdots 2 \cdots (n-1)n, i324 \cdots 1 \cdots (n-1)n, 1i24 \cdots 3 \cdots (n-1)n\}\end{aligned}$$

for $5 \leq i \leq n$, and

$$\begin{aligned}N_0 = \{&2314 \cdots (n-1)n, 4132 \cdots (n-1)n, 3241 \cdots (n-1)n, \\ &4321 \cdots (n-1)n, 1423 \cdots (n-1)n\}.\end{aligned}$$

Also, we have $|N(V(S))| = 6n - 19$. Let $F_1 = N(V(S))$ and $F_2 = N[V(S)]$. We have

$$|F_1| = 6n - 19, |F_2| = 6n - 16,$$

and each component in $AG_n - N[V(S)]$ has more than two vertices by Lemma 3 (2). Hence, $N(V(S))$ is a 2-extra vertex-cut of AG_n. Therefore,

$$|F_1|, |F_2| \leq 6n - 16,$$

and both F_1 and F_2 are 2-extra vertex-sets of AG_n.

Since $F_1 \triangle F_2 = V(S)$, there is no edge between $F_1 \triangle F_2$ and $V(AG_n - F_1 - F_2)$ (see Fig. 5). By Lemma 1, F_1 and F_2 are indistinguishable under the PMC model. By Definition 1, $\widetilde{t_2}^p(AG_n) \leq 6n - 17$.

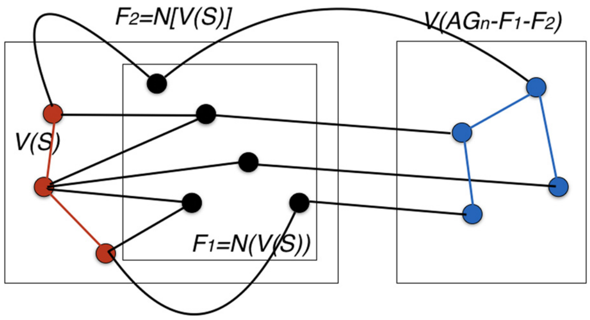

Fig. 5. An illustration that there is no edge between $F_1 \triangle F_2$ and $V(AG_n - F_1 - F_2)$ when $|V(S)| = 3$.

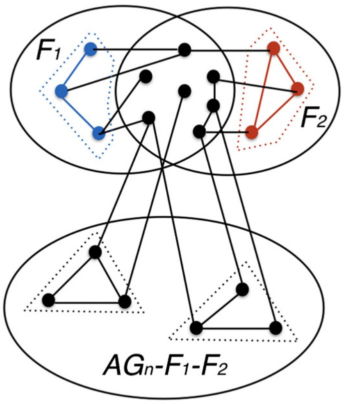

Fig. 6. An illustration that $AG_n - (F_1 \cap F_2)$ is disconnected and every component contains more than two vertices.

Next, we prove the lower bound of 2-extra diagnosability of AG_n by contradiction. Suppose that $\widetilde{t_2}^p(AG_n) \le 6n - 18$. Let F_1 and F_2 be two distinct 2-extra vertex-sets such that (F_1, F_2) is an indistinguishable pair with $|F_1|, |F_2| \le 6n - 17$.

Similar to the proof of Theorem 1, we can deduce that $V(AG_n - F_1 - F_2) \neq \emptyset$ and the vertices of $F_1 \triangle F_2$ have no neighbors outside of $F_1 \cup F_2$ and the vertices of $V(AG_n - F_1 - F_2)$ have no neighbors in $F_1 \triangle F_2$. By the fact that F_1 and F_2 are two distinct 2-extra vertex-sets, every component in $AG_n - F_1$ and $AG_n - F_2$ contains more than two vertices. When $F_1 \cap F_2$ is deleted, $AG_n - (F_1 \cap F_2)$ is disconnected and every component contains more than two vertices (see Fig. 6). By Definition 4 (1), $F_1 \cap F_2$ is a 2-extra vertex-cut of AG_n. By Lemma 3 (2), we have that $|F_1 \cap F_2| \ge 6n - 19$. Since $F_1 \neq F_2$, by the symmetry of AG_n, assume that $F_1 - F_2 \neq \emptyset$. Since F_2 is a 2-extra vertex-set of AG_n, every component of $AG_n - F_2$ contains more than two vertices. Since $E[F_1 \triangle F_2, V(AG_n - F_1 - F_2)] = \emptyset$, every component of $AG_n[F_1 - F_2]$ contains more than two vertices. That is $|F_1 - F_2| \ge 3$. Then

$$\begin{aligned} |F_1| &= |F_1 - F_2| + |F_1 \cap F_2| \\ &\ge 3 + 6n - 19 \\ &= 6n - 16, \end{aligned}$$

which contradicts that $|F_1| \le 6n - 17$.

Hence, $\widetilde{t_2}^p(AG_n) = 6n - 17$. □

Theorem 3. *Let AG_n $(n \ge 5)$ be the n-dimensional alternating group graph. The 3-extra diagnosability of AG_n under the PMC model is $\widetilde{t_3}^p(AG_n) = 8n - 25$.*

Proof. We first prove that $\widetilde{t_3}^p(AG_n) \le 8n - 25$. Let $S = \{1234\cdots i \cdots (n-1)n,\ 3124\cdots i \cdots (n-1)n,\ 4321\cdots i \cdots (n-1)n,\ 2431\cdots i \cdots (n-1)n\}$ be a 4-cycle in AG_n $(n \ge 5)$. We have

$$N(V(S)) = (\bigcup_{5 \le i \le n} N_i) \bigcup N_0,$$

where

$$\begin{aligned} N_i = \{&2i34\cdots 1 \cdots (n-1)n, i134\cdots 2\cdots (n-1)n, i231\cdots 4\cdots (n-1)n, \\ &4i31\cdots 2\cdots (n-1)n, i324\cdots 1\cdots (n-1)n, 1i24\cdots 3\cdots (n-1)n, \\ &3i21\cdots 4\cdots (n-1)n, i421\cdots 3\cdots (n-1)n\} \end{aligned}$$

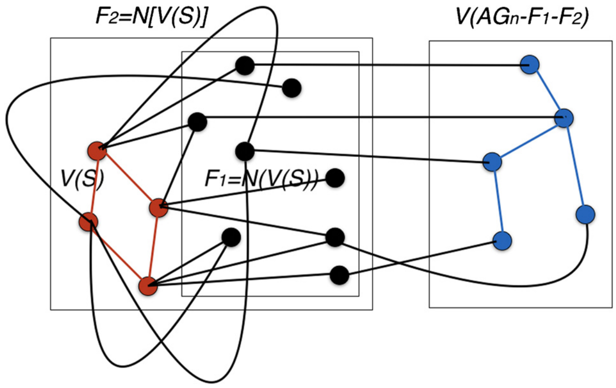

Fig. 7. An illustration that there is no edge between $F_1 \triangle F_2$ and $V(AG_n - F_1 - F_2)$ when $|V(S)| = 4$.

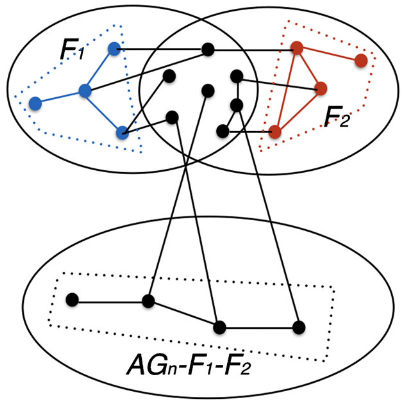

Fig. 8. An illustration that $AG_n - (F_1 \cap F_2)$ is disconnected and every component contains more than three vertices.

for $5 \le i \le n$, and

$$N_0 = \{4132\cdots(n-1)n, 2314\cdots(n-1)n, 3241\cdots(n-1)n, 1423\cdots(n-1)n\}.$$

Also, we have $|N(V(S))| = 8n - 28$. Let $F_1 = N(V(S))$ and $F_2 = N[V(S)]$. Hence,

$$|F_1| = 8n - 28, |F_2| = 8n - 24$$

and each component in $AG_n - N[V(S)]$ has more than three vertices by Lemma 3 (3). Hence, $N(V(S))$ is a 3-extra vertex-cut of AG_n. Therefore,

$$|F_1|, |F_2| \le 8n - 24$$

and both F_1 and F_2 are 3-extra vertex-sets of AG_n.

Since $F_1 \triangle F_2 = V(S)$, there is no edge between $F_1 \triangle F_2$ and $V(AG_n - F_1 - F_2)$ (see Fig. 7). By Lemma 1, F_1 and F_2 are indistinguishable of AG_n under the PMC model. By Definition 1, $\widetilde{t_3}^p(AG_n) \le 8n - 25$.

Next, we prove the lower bound of 3-extra diagnosability of AG_n by contradiction. Suppose that $\widetilde{t_3}^p(AG_n) \le 8n - 26$. Let F_1 and F_2 be two distinct 3-extra vertex-sets such that (F_1, F_2) is an indistinguishable pair with $|F_1|, |F_2| \le 8n - 25$.

Similar to the proof of Theorem 1, we can deduce that $V(AG_n - F_1 - F_2) \ne \emptyset$ and the vertices of $F_1 \triangle F_2$ have no neighbors outside of $F_1 \cup F_2$ and the vertices of $V(AG_n - F_1 - F_2)$ have no neighbors in $F_1 \triangle F_2$. By the fact that F_1 and F_2 are two distinct 3-extra vertex-sets, every component in $AG_n - F_1$ and $AG_n - F_2$ contains more than three vertices. When $F_1 \cap F_2$ is deleted, $AG_n - (F_1 \cap F_2)$ is disconnected and every component contains more than three vertices (see Fig. 8). By Definition 4 (1), $F_1 \cap F_2$ is a 3-extra vertex-cut of AG_n. By Lemma 3 (3), we have that $|F_1 \cap F_2| \ge 8n - 28$. Since $F_1 \ne F_2$, by the symmetry of AG_n, assume that $F_1 - F_2 \ne \emptyset$. Since F_2 is a 3-extra vertex-set of AG_n, every component of $AG_n - F_2$ contains more than three vertices. Since $E[F_1 \triangle F_2, V(AG_n - F_1 - F_2)] = \emptyset$, every component of $AG_n[F_1 - F_2]$ contains more than three vertices. That is $|F_1 - F_2| \ge 4$. Then

$$\begin{aligned}|F_1| &= |F_1 - F_2| + |F_1 \cap F_2| \\ &\geq 4 + 8n - 28 \\ &= 8n - 24,\end{aligned}$$

which contradicts that $|F_1| \leq 8n - 25$.

Hence, $\widetilde{t_3}^p(AG_n) = 8n - 25$. □

4. The g-good-neighbor diagnosability of AG_n under the PMC model

In this section, we will determine the g-good-neighbor diagnosability of AG_n under the PMC model, making use of the following fault tolerance properties of the AG_n.

Lemma 4. [4,8] *Let D be a subset of $V(AG_n)$ $(n \geq 5)$ such that $|D| \leq 4n - 11$. Then $AG_n - D$ satisfies one of the following conditions.*

(1) $AG_n - D$ is connected;

(2) $AG_n - D$ has two components, one of which is a singleton;

(3) $AG_n - D$ has two components, one of which is an edge. Furthermore, $|D| = 4n - 11$ and D is formed by the neighbors of the edge.

Lemma 5. [8] *Let D be a subset of $V(AG_n)$ $(n \geq 5)$ such that $|D| \leq 6n - 19$. Then, $AG_n - D$ satisfies one of the following conditions.*

(1) $AG_n - D$ is connected;

(2) $AG_n - D$ has two components, one of which is a singleton, an edge or a 2-path;

(3) $AG_n - D$ has three components, two of which are both singletons, respectively.

Theorem 4. *Let AG_n $(n \geq 5)$ be an n-dimensional alternating group graph. Then the 1-restricted connectivity is $\kappa^1(AG_n) = 4n - 11$. Furthermore, let $S = uv$ be an edge in AG_n $(n \geq 5)$ such that $u = 1234 \cdots i \cdots n$ and $v = 2431 \cdots i \cdots n$. It can be deduced that $N(V(S))$ is a 1-restricted vertex-cut of AG_n.*

Proof. First, we prove that $\kappa^1(AG_n) \geq 4n - 11$. Suppose that $\kappa^1(AG_n) < 4n - 11$. Let F be a minimum 1-restricted vertex-cut of AG_n. Then $|F| = \kappa^1(AG_n) < 4n - 11$. By Lemma 4, $AG_n - F$ has two components: a singleton and a large component. It contradicts that F is a 1-restricted vertex-cut of AG_n. Therefore, $\kappa^1(AG_n) \geq 4n - 11$.

Next, we prove that the upper bound of $\kappa^1(AG_n)$ is $4n - 11$. Let $S = uv$ be an edge of AG_n such that $u = 1234 \cdots i \cdots n$ and $v = 2431 \cdots i \cdots n$. We have

$$N(V(S)) = (\bigcup_{5 \leq i \leq n} N_i) \bigcup N_0,$$

where

$$\begin{aligned}N_i = \{&2i34 \cdots 1 \cdots (n-1)n, i134 \cdots 2 \cdots (n-1)n, \\ &i231 \cdots 4 \cdots (n-1)n, 4i31 \cdots 2 \cdots (n-1)n\}\end{aligned}$$

for $5 \leq i \leq n$, and

$$\begin{aligned}N_0 = \{&2314 \cdots (n-1)n, 4132 \cdots (n-1)n, 3241 \cdots (n-1)n, \\ &3124 \cdots (n-1)n, 4321 \cdots (n-1)n\}.\end{aligned}$$

Also, we have $|N(V(S))| = 4n - 11$.

We will prove that $N(V(S))$ is a 1-restricted vertex-cut of AG_n. Since $uv \in E(AG_n^n)$, by Lemma 2 (2), u has two external neighbors in different subgraphs AG_n^1 and AG_n^2, and v has two external neighbors in different subgraphs AG_n^2 and AG_n^4. Hence, there exist at most two vertices of AG_n^j $(j \in \{1, 2, 4\})$ in $N(V(S))$. By Lemma 2 (1), $\kappa(AG_{n-1}) = 2n - 6 > 2$ for $n \geq 5$. Hence, $AG_n^j - N_{AG_n^j}(V(S))$ $(j \in \{1, 2, 4\})$ is connected. By Lemma 2 (3), there are $(n-2)!$ disjointed edges between $AG_n^{t_1}$ and $AG_n^{t_2}$ for $t_1, t_2 \in \langle n \rangle - \{n\}$ and $t_1 \neq t_2$. Since $(n-2)! > 4$ for $n \geq 5$, $AG_n^{t_1} - N_{AG_n^{t_1}}(V(S))$ connects $AG_n^{t_2} - N_{AG_n^{t_2}}(V(S))$. By the arbitrariness of $t_1, t_2 \in \langle n \rangle - \{n\}$ and $t_1 \neq t_2$, $AG_n^{\langle n \rangle - \{n\}} - \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(S))$ is connected (see Fig. 9).

Let $x \in V(AG_n^n - N_{AG_n^n}[V(S)])$. By Lemma 2 (2), x has two external neighbors x' and x''. By Lemma 2 (2), x' and x'' are not in $\bigcup_{j=1}^{n-1} N_{AG_n^j}(V(S))$. (Otherwise, assume that $x' \in \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(S))$. Hence, x' has two external neighbors $y \in V(S)$ in subgraph AG_n^n and $x \in V(AG_n^n - N[V(S)])$ in subgraph AG_n^n. It implies that x' has two external neighbors in the same subgraph, which contradicts to Lemma 2 (2).) Thus, x' and x'' are in $AG_n^{\langle n \rangle - \{n\}} - \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(S))$. Hence, x is connected to $AG_n^{\langle n \rangle - \{n\}} - \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(S))$. By the arbitrariness of $x \in V(AG_n^n - N[V(S)])$, $AG_n - N[V(S)]$ is connected. Therefore, $AG_n - N(V(S))$ has two components: $AG_n - N[V(S)]$ and S such that $\delta(AG_n - N[V(S)]) \geq 1$ and $\delta(S) \geq 1$. Hence, $N(V(S))$ is a 1-restricted vertex-cut of AG_n by Definition 4 (2). By Definition 4 (2), $\kappa^1(AG_n) \leq |N(V(S))| = 4n - 11$.

Therefore, $\kappa^1(AG_n) = 4n - 11$ for $n \geq 5$ and $N(V(S))$ is a 1-restricted vertex-cut of AG_n. □

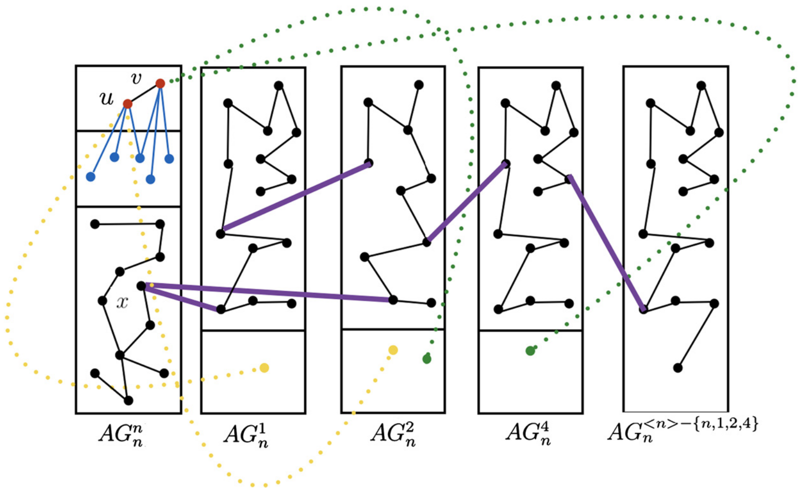

Fig. 9. An illustration that $AG_n^{\langle n\rangle-\{n\}} - \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(S))$ is connected.

Theorem 5. *Let AG_n $(n \geq 5)$ be an n-dimensional alternating group graph. Then, the 2-restricted connectivity is $\kappa^2(AG_n) = 6n - 18$. Furthermore, let $C_3 = \{u, v, w\}$ be a 3-cycle in AG_n $(n \geq 5)$ such that $u = 1234\cdots i\cdots n$, $v = 2431\cdots i\cdots n$ and $w = 4132\cdots i\cdots n$. It can be deduced that $N(V(C_3))$ is a 2-restricted vertex-cut of AG_n.*

Proof. First, we prove that $\kappa^2(AG_n) \geq 6n-18$. Suppose that $\kappa^2(AG_n) < 6n-18$. Let F be a minimum 2-restricted vertex-cut of AG_n, $|F| = \kappa^2(AG_n) < 6n - 18$. By Lemma 5, $AG_n - F$ has a singleton and a large component; or has an edge and a large component; or has a 2-path and a large component; or has two singletons and a large component. It contradicts that F is a 2-restricted vertex-cut of AG_n. Hence, $\kappa^2(AG_n) \geq 6n - 18$.

Next, we prove that $\kappa^2(AG_n) \leq 6n - 18$. Let $C_3 = \{u, v, w\}$ be a 3-cycle in AG_n such that $u = 1234\cdots i\cdots n$, $v = 2431\cdots i\cdots n$ and $w = 4132\cdots i\cdots n$. Obviously, C_3 is in AG_n^n. Furthermore, we have

$$N(V(C_3)) = (\bigcup_{5\leq i\leq n} N_i)\bigcup N_0,$$

where

$$N_i = \{2i34\cdots1\cdots(n-1)n, i134\cdots2\cdots(n-1)n, 4i31\cdots2\cdots(n-1)n, i231\cdots4\cdots(n-1)n, 1i32\cdots4\cdots(n-1)n, i432\cdots1\cdots(n-1)n\}$$

for $5 \leq i \leq n$, and

$$N_0 = \{2314\cdots(n-1)n, 3124\cdots(n-1)n, 3241\cdots(n-1)n, 4321\cdots(n-1)n, 3412\cdots(n-1)n, 1342\cdots(n-1)n\}.$$

Also, we have $|N(V(C_3))| = 6n - 18$.

We will prove that $N(V(C_3))$ is a 2-restricted vertex-cut of AG_n. By Lemma 2 (2), two external neighbors of u are in two subgraphs AG_n^1 and AG_n^2, respectively, two external neighbors of v are in two subgraphs AG_n^4 and AG_n^2, respectively, and two external neighbors of w are in two subgraphs AG_n^1 and AG_n^4, respectively. Hence, there exist at most two vertices of AG_n^j $(j \in \{1, 2, 4\})$, which are in $N(V(C_3))$. By Lemma 2 (1), $\kappa(AG_{n-1}) = 2n - 6 > 2$ for $n \geq 5$. Hence, $AG_n^j - N_{AG_n^j}(V(C_3))$ $(j \in \{1, 2, 4\})$ is connected. By Lemma 2 (3), there are $(n-2)!$ disjointed edges between $AG_n^{t_1}$ and $AG_n^{t_2}$ for $t_1, t_2 \in \langle n\rangle - \{n\}$ and $t_1 \neq t_2$. Since $(n-2)! > 4$ for $n \geq 5$, $AG_n^{t_1} - N_{AG_n^{t_1}}(V(C_3))$ connects $AG_n^{t_2} - N_{AG_n^{t_2}}(V(C_3))$. By the arbitrariness of $t_1, t_2 \in \langle n\rangle - \{n\}$ and $t_1 \neq t_2$, $AG_n^{\langle n\rangle-\{n\}} - \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(C_3))$ is connected (see Fig. 10).

Let $x \in V(AG_n^n - N[V(C_3)])$. By Lemma 2 (2), x has two external neighbors x' and x''. By Lemma 2 (2), both x' and x'' are not in $\bigcup_{j=1}^{n-1} N_{AG_n^j}(V(C_3))$. (Otherwise, assume that $x' \in \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(C_3))$. Hence, x' has two external neighbors $y \in V(C_3)$ in subgraph AG_n^n and $x \in V(AG_n^n - N[V(C_3)])$ in subgraph AG_n^n. It implies that x' has two external neighbors in the same subgraph, which contradicts to Lemma 2 (2).) Thus, x' and x'' are in $AG_n^{\langle n\rangle-\{n\}} - \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(C_3))$. Hence, x is connected to $AG_n^{\langle n\rangle-\{n\}} - \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(C_3))$. By the arbitrariness of $x \in V(AG_n^n - N[V(C_3)])$, $AG_n - N[V(C_3)]$ is connected. Next, we need to prove that $|N_{AG_n-N[V(C_3)]}(x)| \geq 2$ for any $x \in V(AG_n - N[V(C_3)])$. If $x \in V(AG_n^n - N[V(C_3)])$, then x has two external neighbors x' and x'' in $V(AG_n^{\langle n\rangle-\{n\}} - \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(C_3)))$. Hence, $|N_{AG_n-N[V(C_3)]}(x)| \geq |N_{AG_n^{\langle n\rangle-\{n\}}-\bigcup_{j=1}^{n-1} N_{AG_n^j}(V(C_3))}(x)| \geq 2$. If $x \in V(AG_n^j - N_{AG_n^j}(V(C_3)))$ for any $j \in \{1, 2, 4\}$, then x has at least $2n - 6 - 2 \geq 2$ neighbors in $V(AG_n^j - N_{AG_n^j}(V(C_3)))$ for

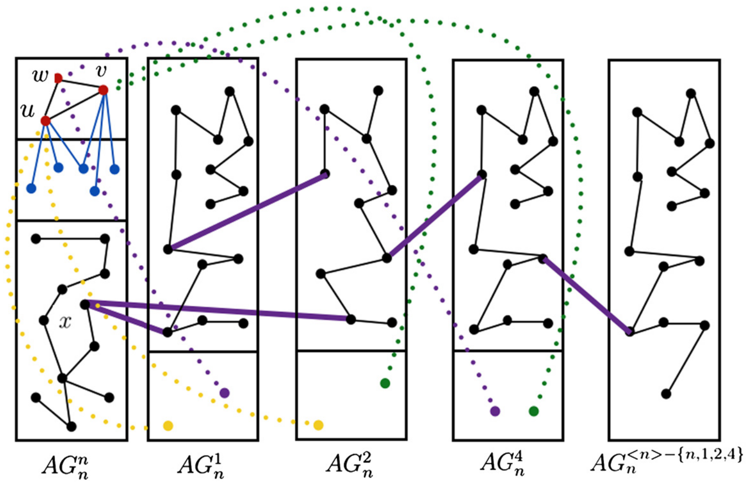

Fig. 10. An illustration that $AG_n^{\langle n\rangle-\{n\}} - \bigcup_{j=1}^{n-1} N_{AG_n^j}(V(C_3))$ is connected.

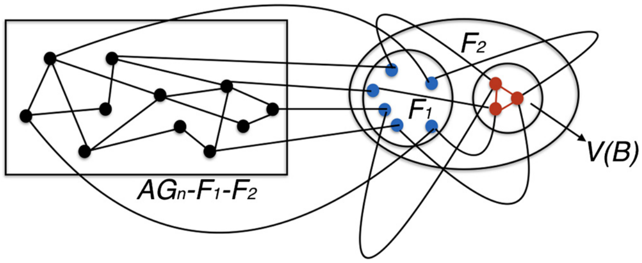

Fig. 11. An illustration that (F_1, F_2) is an indistinguishable pair of AG_n under the PMC model.

$n \geq 5$ by Lemma 2 (1). Hence, $|N_{AG_n-N[V(C_3)]}(x)| \geq |N_{AG_n^j-N_{AG_n^j}(V(C_3))}(x)| \geq 2$. If $x \in V(AG_n^i)$ for any $i \in \langle n\rangle - \{n, 1, 2, 4\}$, then x has $2n-6 \geq 4$ neighbors in $V(AG_n^i - N_{AG_n^i}(V(C_3)))$ for $n \geq 5$ by Lemma 2 (1). Hence, $|N_{AG_n-N[V(C_3)]}(x)| \geq |N_{AG_n^i-N_{AG_n^i}(V(C_3))}(x)| \geq 2$. By the arbitrariness of $j \in \{1, 2, 4\}$ and $i \in \langle n\rangle - \{n, 1, 2, 4\}$, $|N_{AG_n-N[V(C_3)]}(x)| \geq 2$ for any $x \in V(AG_n - N[V(C_3)])$. Therefore, $AG_n - N(V(C_3))$ has two components: $AG_n - N[V(C_3)]$ and C_3 such that $\delta(AG_n - N[V(C_3)]) \geq 2$ and $\delta(C_3) \geq 2$. Hence, by Definition 4 (2), $N(V(C_3))$ is a 2-restricted vertex-cut of AG_n. By Definition 4 (2), $\kappa^2(AG_n) \leq |N(V(C_3))| = 6n-18$. Therefore, $\kappa^2(AG_n) = 6n-18$ for $n \geq 5$ and $N(V(C_3))$ is a 2-restricted vertex-cut of AG_n. □

To find the g-good-neighbor diagnosability $t_g^p(AG_n)$ under the PMC model, we first show, by construction, that $t_g^p(AG_n)$ is no more than $(2g+2)n - 2^{g+2} - 4 + 2g$ for $1 \leq g \leq 2$.

Theorem 6. *Let AG_n ($n \geq 5$) be an n-dimensional alternating group graph. The upper bound of g-good-neighbor diagnosability of AG_n under the PMC model is $(2g+2)n - 2^{g+2} - 4 + 2g$ for $1 \leq g \leq 2$.*

Proof. Let $B \cong K_{g+1}$ be a complete subgraph of AG_n for $1 \leq g \leq 2$. Let $F_1 = N(V(B))$ and $F_2 = N[V(B)]$ (see Fig. 11). By Theorem 4 and Theorem 5,

$$|F_1| = (2g+2)n - 2^{g+2} - 4 + g, |F_2| = (2g+2)n - 2^{g+2} - 3 + 2g,$$

and $\delta(AG_n - F_2) \geq g$. By Definition 2 (1), F_1 and F_2 are two g-good-neighbor vertex-sets of $V(AG_n)$ with

$$|F_1| \leq (2g+2)n - 2^{g+2} - 3 + 2g, |F_2| \leq (2g+2)n - 2^{g+2} - 3 + 2g.$$

On the other hand, since $V(B) = F_1 \triangle F_2$, $N(V(B)) = F_1$ and $F_1 \subset F_2$, there is no edge between $V(AG_n - F_1 - F_2)$ and $F_1 \triangle F_2$. By Lemma 1, (F_1, F_2) is an indistinguishable pair of AG_n under the PMC model. By Definition 2 (2), the n-dimensional alternating group graph AG_n is not g-good-neighbor $[(2g+2)n - 2^{g+2} - 3 + 2g]$-diagnosable under the PMC model. By Definition 2 (3), the upper bound of g-good-neighbor diagnosability of AG_n is

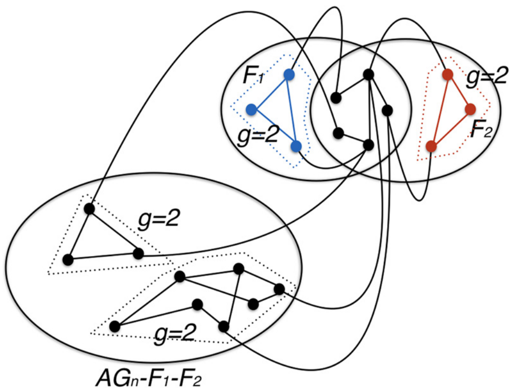

Fig. 12. An illustration that $F_1 \cap F_2$ is a g-good-neighbor vertex-set in AG_n.

$$t_g^p(AG_n) \leq (2g+2)n - 2^{g+2} - 4 + 2g$$

under the PMC model for $1 \leq g \leq 2$. Hence, Theorem 6 holds. □

Theorem 7. *Let AG_n $(n \geq 5)$ be an n-dimensional alternating group graph. The lower bound of g-good-neighbor diagnosability of AG_n under the PMC model is $(2g+2)n - 2^{g+2} - 4 + 2g$ for $1 \leq g \leq 2$.*

Proof. By Definition 2 (3), we just need to show that AG_n is g-good-neighbor $[(2g+2)n - 2^{g+2} - 4 + 2g]$-diagnosable. By Definition 2 (2), to prove that AG_n is g-good-neighbor $[(2g+2)n - 2^{g+2} - 4 + 2g]$-diagnosable, it is equivalent to prove that for every two distinct g-good-neighbor vertex-sets F_1 and F_2 of $V(AG_n)$ with

$$|F_1| \leq (2g+2)n - 2^{g+2} - 4 + 2g \text{ and } |F_2| \leq (2g+2)n - 2^{g+2} - 4 + 2g,$$

F_1 and F_2 must be distinguishable.

We prove this statement by contradiction. Suppose that there are two distinct g-good-neighbor vertex-sets F_1 and F_2 with $|F_1| \leq (2g+2)n - 2^{g+2} - 4 + 2g$ and $|F_2| \leq (2g+2)n - 2^{g+2} - 4 + 2g$, but (F_1, F_2) is an indistinguishable pair. Now we consider all the possible cases such that F_1 and F_2 are indistinguishable. By Lemma 1, we have $V(AG_n) = F_1 \cup F_2$ or $V(AG_n) \neq F_1 \cup F_2$ but there exists no edge between $V(AG_n - F_1 - F_2)$ and $F_1 \triangle F_2$. By the symmetry of AG_n, assume that $F_2 - F_1 \neq \emptyset$. We will show that each of the following cases deduces a contradiction against our assumption.

Case 1. $V(AG_n) = F_1 \cup F_2$**.**

For $n \geq 5$, $1 \leq g \leq 2$ and $V(AG_n) = F_1 \cup F_2$, we have

$$\frac{n!}{2} = |V(AG_n)| > 2[(2g+2)n - 2^{g+2} - 4 + 2g] \geq |F_1| + |F_2| \geq |V(AG_n)|,$$

which is a contradiction.

Case 2. $V(AG_n) \neq F_1 \cup F_2$ **but there exists no edge between** $V(AG_n - F_1 - F_2)$ **and** $F_1 \triangle F_2$**.**

By the assumption that $F_2 - F_1 \neq \emptyset$ and F_1 is a g-good-neighbor vertex-set, any vertex in $F_2 - F_1$ has at least g good neighbors in $AG_n[F_2 - F_1]$. We have $|F_2 - F_1| \geq g + 1$. Since F_1 and F_2 are both g-good-neighbor vertex-sets, $F_1 \cap F_2$ is also a g-good-neighbor vertex-set (see Fig. 12). In addition, since there are no edges between $V(AG_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $AG_n - (F_1 \cap F_2)$ is disconnected and any component of $AG_n - (F_1 \cap F_2)$ has the minimum degree g. Hence, $F_1 \cap F_2$ is a g-restricted vertex-cut of AG_n. By Theorem 4 and Theorem 5,

$$|F_1 \cap F_2| \geq (2g+2)n - 2^{g+2} - 4 + g.$$

Therefore, we have

$$\begin{aligned} |F_2| &= |F_2 - F_1| + |F_1 \cap F_2| \\ &\geq g + 1 + (2g+2)n - 2^{g+2} - 4 + g \\ &= (2g+2)n - 2^{g+2} - 3 + 2g, \end{aligned}$$

which contradicts against $|F_2| \leq (2g+2)n - 2^{g+2} - 4 + 2g$.

Based on the above discussion, we conclude that $t_g^p(AG_n) \geq (2g+2)n - 2^{g+2} - 4 + 2g$. Hence, Theorem 7 holds. □

Combining Theorem 6 and Theorem 7, we have the following theorem.

Table 1
Extra and good-neighbor diagnosability of AG_n $(n \geq 5)$ under PMC model.

Number	$h=1$	$h=2$	$h=3$
Extra connectivity [21]	$\kappa_0^{(1)}(AG_n)=4n-11$	$\kappa_0^{(2)}(AG_n)=6n-19$	$\kappa_0^{(3)}(AG_n)=8n-28$
Extra diagnosability	$\widetilde{t_1}^p(AG_n)=4n-10$	$\widetilde{t_2}^p(AG_n)=6n-17$	$\widetilde{t_3}^p(AG_n)=8n-25$

Table 2
Good-neighbor diagnosability of AG_n $(n \geq 5)$ under PMC model.

Number	$g=1$	$g=2$
Restricted connectivity	$\kappa^1(AG_n)=4n-11$	$\kappa^2(AG_n)=6n-18$
Good-neighbor diagnosability	$t_1^p(AG_n)=4n-10$	$t_2^p(AG_n)=6n-16$

Theorem 8. *Let AG_n $(n \geq 5)$ be an n-dimensional alternating group graph. Let $t_g^p(AG_n)$ denote the g-good-neighbor diagnosability of AG_n under the PMC model. We have $t_g^p(AG_n)=(2g+2)n-2^{g+2}-4+2g$ for $1 \leq g \leq 2$.*

Furthermore, we add two tables (see Table 1 and Table 2) to summarize the main results in order to make them more clearly.

These two tables reveal the relationship between extra (resp., restricted) connectivity and extra (resp., good-neighbor) diagnosability of AG_n, which is a regular graph with 3-cycle.

5. Conclusion

In this paper, we first establish that the 1-extra diagnosability of AG_n under the PMC model is $4n-10$ for $n \geq 5$. Then we prove that the 2-extra diagnosability of AG_n under the PMC model is $6n-17$ for $n \geq 5$. Next, we address that the 3-extra diagnosability of AG_n under the PMC model is $8n-25$ for $n \geq 5$. Finally, we obtain that the g-restricted connectivity and the g-good-neighbor diagnosability of AG_n $(n \geq 5)$ are $(2g+2)n-2^{g+2}-4+g$ and $(2g+2)n-2^{g+2}-4+2g$ for $1 \leq g \leq 2$, respectively.

In future, we will apply the h-extra and g-good-neighbor diagnosability into mobile social networks (MSNs). In particular, we will consider the detection of malicious users in MSNs.

Declaration of Competing Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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